Exercise Sheet 5 due 20 November 2014

1. *Hermite polynomials*

You can represent a polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$ of order smaller than n by an n-dimensional array `a[i]`. Write a code that calculates the lowest 20 Hermite polynomials $H_n(x)$ from the recursion relation

$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$$

starting from $H_0(x) = 1$ and $H_1(x) = 2x$.

Your code should be able to print the polynomials and to evaluate them numerically so you can produce a plot.

2. *harmonic oscillation*

Consider a harmonic oscillator with Hamiltonian

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}\, x^2$$

with eigenfunctions $H\phi_n = \hbar\omega(n + 1/2)\,\phi_n(x)$. For a wave packet at time $t = 0$ given by

$$\Psi(x, t = 0) = \sum_n c_n\, \phi_n(x)$$

show that the expectation value of the position of the electron

$$x(t) = \int dx\, \overline{\Psi(x, t)}\, x\, \Psi(x, t)$$

oscillates harmonically with frequency ω: $x(t) = x_0 \cos(\omega t + \delta)$.
Express x_0 and δ in terms of the amplitudes c_n at $t = 0$.

3. *ladder operators*

Consider the harmonic Hamiltonian $H = a^\dagger a\ + 1/2$ with operators $a = \left(\zeta + \frac{d}{d\zeta}\right)/\sqrt{2}$ and $a^\dagger = \left(\zeta - \frac{d}{d\zeta}\right)/\sqrt{2}$.

i. Use integration by parts (twice) to show that

$$\langle\phi_n|H\phi_m\rangle = \int d\zeta\, \overline{\phi_n(\zeta)}\, (H\,\phi_m(\zeta)) = \int d\zeta\, (\overline{H\,\phi_n(\zeta)})\, \phi_m(\zeta) = \overline{\langle H\phi_m|\phi_n\rangle}$$

ii. Show that the eigenfunctions with different eigenenergies are orthogonal.

iii. Show that $\zeta = (a\ + a^\dagger)/\sqrt{2}$.

iv. Show that for the eigenfunctions $H|\phi_n\rangle = (n + 1/2)\,|\phi_n\rangle$ holds

$$\langle\phi_n|\zeta|\phi_m\rangle = \int d\zeta\, \phi_n(\zeta)\, \zeta\, \phi_m(\zeta) = \sqrt{\frac{n+1}{2}}\,\delta_{n,m-1} + \sqrt{\frac{n}{2}}\,\delta_{n,m+1}$$